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A continuum theory for the flow of pedestrians

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Abstract

The equations of motion governing the two-dimensional flow of pedestrians are derived for flows of both single and multiple pedestrian types. Two regimes of flow, a high-density (subcritical) and a low-density (supercritical) flow regimes, are possible, rather than two flow regimes for each type of pedestrian. A subcritical flow always fills the space available. However, a supercritical flow may either fill the space available or be self-confining for each type of pedestrian, depending on the boundary location. Although, the equations governing these flows are simultaneous, time-dependent, non-linear, partial differential equations, remarkably they may be made conformally mappable. The solution of these equations becomes trivial in many situations. Free streamline calculations, utilizing this property, reveal both upstream and downstream separation of the flow of pedestrians around an obstacle. Such analysis tells much about the nature of the assumptions used in various models for the flow of pedestrians. The present theory is designed for the development of general techniques to understand the motion of large crowds. However, it is also useful as a predictive tool. The behavior predicted by these equations of motion is compared with aerial observations for the Jamarat Bridge near Mecca, Saudi Arabia. It is shown that, for this important case, pedestrians, that is pilgrims, aim at achieving each immediate goal in minimum time rather than achieving all goals in overall minimum time. Typical of many examples, this case illustrated the strong dependence of path on the psychological state of the pedestrians involved. It is proposed that the flow of pedestrians over the Jamarat Bridge be improved by appropriate barrier placement, that force an effective global view of the goals.

1. Introduction

The flow of large crowds of pedestrians is likely to become increasingly important as the populations of our large cities grow. Many studies of pedestrian flows have been undertaken

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particularly over the past three decades, with particularly strong interest in the topic since the early 1990s, testifying to its currently perceived importance, (Smith and Dickie, 1993). Nevertheless, as Wigan (1993) has noted, our knowledge of the flow of crowds is inadequate and behind that of other transport modes. Interest in crowd modeling has many sources. Examples include crowds associated with transport systems (Daly et al., 1991; Toshiyuki, 1993; Smith, 1993), sporting and general spectator occasions (Bradley, 1993), holy sites (Al Gadhi and Mahmassani, 1991; Selim and Al-Rabeh, 1991), political demonstrations (Surti and Burke, 1971) and fire escapes (Tanaka, 1991). No attempt is made here to present a complete list of situations. However, the above studies illustrate the varied behavior associated with different situations. The behavior of pedestrians varies not only with their physical characteristics but also with their purpose as shown by Polus et al. (1983), and Pushkarev and Zupan (1975).

There are two fundamentally distinct philosophies for modeling crowd motion. The first philosophy involves treating pedestrians as discrete individuals and walks them through the domain generally in a computer simulation. Pedestrians are modeled by (a) using a granular material analogue (rare), (b) modeling the path taken assuming pedestrians optimize their immediate local behavior, or (c) assuming they attempt to move along predefined globally determined paths. Both Lagrangian simulation, whereby individuals are followed through the domain, and Eulerian simulation, whereby an account is kept of the number of individuals in each grid box in the domain, are used. The second philosophy, applicable only in large crowds, involves treating the crowd as a whole. Crowds are treated as (a) a fluid (now rare), (b) a continuum responding to local influences, or (c), as in this study, by assuming individuals in the continuum move so as to optimize their behavior to reach non-local objectives. To the author's knowledge when using the second philosophy attention has been restricted to Eulerian modeling. The alternative Lagrangian theory may be developed quite simply by following the procedure in Lamb (1932).

The former modeling philosophy provides flexibility. It is particularly well suited to be used for small crowds. The latter philosophy is probably better in understanding the rules governing the overall behavior of these flows. However, as a modeling tool this latter philosophy excels with extremely large crowds especially in studying those aspects of the motion for which individual differences are not important. Thus, a continuum model alleviates a major problem in understanding crowds as noted by Gaskell and Benewick (1987), namely, that a crowd needs to be treated as an identity. Both philosophies have their place. They should be seen as complementing each other.

The applicability of the various techniques is also varied. The use of (granular and) fluid analogies is of limited value except at dangerously high-crowd densities ($>4\ \mathrm{m}^{-2}$) where (compressibility) pressure waves can occur. Unlike a material substance, at lower densities, a crowd does not accelerate but walks at constant speed when subjected to a motivational force. Consistent with this idea Bradley (1993) hypothesized that the Navier–Stokes equations governing fluid motion could be used to describe motion in crowds at very high densities. However, a different fluid analogy was used by Henderson and Jenkins (1974) and Henderson (1974). They sought to explain pedestrian motion by analogy with a molecular fluid. However, the molecular approach is cumbersome at these high densities.

With regard to the non-fluid analog methods, whether the local or non-local rules of behavior are assumed by the modeler is a matter of how the modeler perceives the crowd. A crowd is a hybrid of individual perceptions and as such the true nature of the flow possibly lies somewhere between the predictions of the various modeling approaches. What is important is not what model type is used but that any significant conclusions are either not model dependent or if dependent that they are fully understood.

The purpose of the present study is to develop a theoretical framework for understanding the mechanics of pedestrian crowd motion especially large crowds. The primary purpose is to understand what principles underlie the motion of pedestrian crowds by a two-dimensional generalization of the behavior found by Lighthill and Whitham (1955) (see also Whitham, 1974) for vehicle motion. The Lighthill and Whitham (1955) theory is a continuum theory in which traffic density is considered rather than discrete vehicles. As such, it suffers from a lack of realism at very low traffic densities unless it is interpreted probabilistically. Furthermore, it is not applicable to flows that are changing on a time-scale less than the response time of a driver. However, it is extremely robust and gives realistic representation of most vehicular flows, other than those at very low traffic densities.

To the author's knowledge, in all quantitative models of the motion of a crowd the members of the crowd have been modeled as having a well-defined goal at a known location. McFarland (1989) describes such behavior as "goal-directed". Other behavioral types are "goal-seeking" (where the goal is actively sought and achieved without a highly defined strategy) and "goal-identifying" (where the goal is ultimately achieved despite their being no strategy or clear knowledge of what is the goal). Commonly in the literature on animal behavior, goal-directed and goal-seeking are grouped together. The present study continues to restrict attention to goal-directed pedestrians.

To obtain the equations governing a crowd of pedestrians the crowd must behave rationally. It is argued that the individuals in the crowd are guided by reason and rules of behavior. Thus, the crowd is also governed by rules of behavior. (Indeed a crowd of immense size, the "thermodynamic limit", is expected to be better able to be described by deterministic rules of motional behavior than the individuals in it.) Such sentiments are consistent with the modern understanding of crowd behavior as studied for instance by McPhail (1991). (Interestingly, it is commonly stated in the sociological literature, including by McPhail, that the idea that crowds are irrational is a misconception originating with the aristocracy during the French Revolution. Johnson's Dictionary in Miniature, published during the French Revolution, supports this idea and defines the substantive (noun) as 'a confused multitude, the populace' with the 1799 version containing a history of the French Revolution. However, the idea that a crowd is irrational dates back much further. The first edition of Johnson's Dictionary, 1755, thirty five years before the French Revolution, cites Watts for the definition of the verb 'to fill with confused multitudes'.) Thus, we can assume that a crowd behaves rationally. Note that vehicular traffic does not behave in a rational manner with regard its choice of path because it is not fully informed. Pedestrian flows behave in a rational manner because the situation is fully visible to the taller members of the flow, whom it is supposed convey information to the shorter members by their actions.

In the present manuscript, Section 2 deduces, from simple and reasonable hypotheses, the equations of motion for a flow of pedestrians of single pedestrian type. Section 3 shows how the equations derived lead to the two regimes of flow, the subcritical and supercritical regimes, with attention given to the behavior of disturbances in the flow. The theory is extended in Section 4, to crowds composed of multiple types of pedestrians. Section 5 applies the theory to the important example of the flow of pedestrians, in the form of pilgrims, over the Jamarat Bridge near Mecca. This bridge has been the site of some horrific accidents in the past. Section 6 introduces a new alternative relationship for the speed of pedestrians that, as shown in Section 7, has the intriguing property that, despite being non-linear and time-dependent, the coupled equations governing the flow of pedestrians are conformally mappable. This enables the equations to be solved trivially in many situations. In Sections 8–10 conformal mapping is used to study the flow around a pillar with the intent to investigating possible improvements in the flow over the Jamarat Bridge. These sections consider medium-, low- and high-density flows, respectively. The conclusions are given in Section 11.

2. The equations governing pedestrians of a single type

We describe a pedestrian flow, in which only one type of pedestrian is involved, in terms of two qualities. These qualities are:

(i) density, ρ, of the flow, which is the expected number of individuals located within unit area of floor space at a given time, t, and location (x, y), e.g. 0.9 m^{-2}, and
(ii) velocity, (u, v), of the flow, which is the expected velocity of individuals at a given time, t, and location, (x, y), e.g. (0.1, 0.5) m/s.

In the present study it is assumed that any variation from the expected value is negligible and so ρ and (u, v) may be taken as their local mean values. Thus, conservation of pedestrians implies

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \tag{2.1}$$

(The reader unfamiliar with (2.1) will find this equation rigorously derived in texts on fluid mechanics under the name 'the continuity equation'. The equation follows simply by equating the net flow of pedestrians into a small region to the time rate of accumulation of pedestrians in the region, and letting the area of the region shrink to zero.) To proceed further it is necessary to make hypotheses about the nature of pedestrian motion. Three hypotheses are made here.

Hypothesis 1. The speed at which pedestrians walk is determined solely by the density of the surrounding pedestrian flow and the behavioral characteristics of the pedestrians.

Thus for a single type of pedestrian the velocity components are given by

$$u = f(\rho)\hat{\phi}_x \quad v = f(\rho)\hat{\phi}_y, \tag{2.2}$$

where $f(\rho)$ is the speed and $\hat{\phi}_x$ and $\hat{\phi}_y$ are direction cosines of the motion. This hypothesis is standard. For the crowds of interest here where the density is not extreme, although possibly high, the speed of a pedestrian is determined by the density of surrounding pedestrians in a manner similar to Greenshields (1934) model of vehicular flow. This assumption is fundamental to use of the Lighthill and Whitham (1955) model. It has been verified for numerous situations. However, there is no uniformly accepted form of the function relating the speed and the density, nor can there be, as both the psychological state of pedestrians and the state of the ground under foot are very important.

Now in the paper the author does say that the speed depends on the density and behavioral factor. Density with speed is given by f(q). but behavioral relation with speed is not mentioned and the paper suggest can never be found.

Hypothesis 2. Pedestrians have a common sense of the task (called potential) they face to reach their common destination such that any two individuals at different locations having the same potential would see no advantage to either of exchanging places.

There is no perceived advantage of moving along a line of constant potential. Thus, the motion of any pedestrian is in the direction perpendicular to the potential, that is, in the direction for which

$$\hat{\phi}_x = \frac{-\frac{\partial \phi}{\partial x}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}}, \qquad \hat{\phi}_y = \frac{-\frac{\partial \phi}{\partial y}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}}, \tag{2.3}$$

where ϕ is the potential. The derivation of (2.3) may be found in a text on vector algebra under the title unit normal vector or unit gradient vector. This hypothesis is not appropriate to vehicular traffic but appears to be applicable to pedestrian flows where pedestrians can visually assess the situation. Note that for the hypothesis to be valid it must be assumed that shorter pedestrians take their direction from the tallest pedestrians who have an overall view of the situation. This assumption is applicable to most crowds but not to all. However, even when not applicable it generally provides an acceptable approximation.

Hypothesis 3. Pedestrians seek to minimize their (accurately) estimated travel time, but temper this behavior to avoid extremely high densities. This tempering is assumed to be 'separable', such that pedestrians minimize the product of their travel time and a function of the density.

Two pedestrians on a given potential must both be at the same new potential as each other at some later time (noting time is a measure of potential by Hypothesis 3). Thus, the distance between potentials must be proportional to pedestrian speed irrespective of the initial position (on a line of constant potential) of a pedestrian. Thus, we write

$$\frac{1}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}} = g(\rho)\sqrt{u^2 + v^2}, \tag{2.4}$$

where ϕ has been scaled appropriately and $g(\rho)$ is a factor to allow for discomfort at very high densities in accordance with Hypothesis 3. The factor $g(\rho)$ is equal to unity for most densities but rises for high densities. Except where stated it is taken as unity in this study

Eqs. (2.1)–(2.4) combine to form the governing equations for pedestrian flow

$$-\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho g(\rho) f^2(\rho) \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho g(\rho) f^2(\rho) \frac{\partial \phi}{\partial y}\right) = 0 \tag{2.5a}$$

and

$$g(\rho) f(\rho) = \frac{1}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}}. \tag{2.5b}$$

Eq. (2.5a) and (2.5b) requires the specification of boundary conditions in any particular situation. Generally but not always, ρ is specified on those open boundaries that correspond to entrances. Note that by specifying ρ the speed, $f(\rho)$, and the flow, $\rho f(\rho)$, are specified automatically. The potential, ϕ, is zero at exits and the normal derivative of ϕ is specified as zero on closed boundaries. For any slowly moving boundary, such as next to a slowly moving vehicle, the normal components of velocity of both the pedestrians and the boundary must be equal. For a rapidly moving boundary, safety issues are important and the boundary condition depends on psychological influences.

Many physical situations are potentially both anisotropic and inhomogeneous. Fortunately, writing the speed a $f(\rho, \hat{\phi}_x, t, x, y)$ rather than $f(\rho)$, as used in (2.2), does not change the derivation of the equations of motion and Eqs. (2.5a) and (2.5b), with f redefined for the new arguments, remains valid. There are numerous situations where these additional terms are important. However, this extension is not pursued in here.

It is sometimes necessary to introduce additional hypotheses to overcome ambiguities that may pertain to a specific situation. The author refers to any such additional hypotheses as "local hypotheses." The vast majority of situations are covered by Eqs. (2.5a) and (2.5b). In addition, it is occasionally necessary to modify the above hypotheses for local custom. The author refers to such modifications as "local modifications". The role of both local hypotheses and local modifications appears to be of greater importance for smaller crowd sizes where queuing and limited width flows occur as are well illustrated by Kalett (1978).

3. Properties of solutions

The flow of pedestrians per unit width, q, is related to the density of the flow by

$$q = \rho f(\rho). \tag{3.1}$$

Difficulty arises in choosing an appropriate form for $f(\rho)$. The functions, $f(\rho)$ and $g(\rho)$, much have the following properties:

$$\begin{aligned} &\text{(i)}\ f(0)\ \text{is finite,}\quad \text{(ii)}\ f(\rho_{\max}) = 0 \quad \text{and} \quad \text{(iii)}\ \frac{\mathrm{d}f(\rho)}{\mathrm{d}\rho} \leqslant 0, \\ &\text{(iv)}\ g(\rho) \geqslant 1 \quad \text{and} \quad \text{(v)}\ \frac{\mathrm{d}g(\rho)}{\mathrm{d}\rho} \geqslant 0, \end{aligned} \tag{3.2}$$

where $\rho_{\max}$ is the density at which pedlock occurs. The function $g(\rho)$ has generally been taken as unity in the literature. However, there have been several functional forms proposed for $f(\rho)$. The function $f(\rho)$ is taken here, initially, to be of linear form, that is the Greenshields (1934) model,

$$f(\rho) = A - B\rho, \tag{3.3}$$

where A and B are positive constants. Typical values for A and B may be 1.4 m/s and 0.25 m^3/s, respectively, as summarized by Pushkarev and Zupan (1975). Note that the shape of $f(\rho)$ as given here by (3.3) is merely a rough description for pedestrian behavior as studied by earlier authors. However, such a linear variation of $f(\rho)$ with ρ does not capture the constant nature of $f(\rho)$ at small ρ, as seen in observations reported by Toshiyuki (1993), nor does it capture the broad region of constant $\rho f(\rho)$ reported by Smith (1993). Later an alternative form for $f(\rho)$ will be considered which is considered closer to observed behavior and has very useful properties. By (3.1) and (3.3)

$$q = \rho(A - B\rho). \tag{3.4}$$

The flow q has a maximum value of $A^2/4B$ at $\rho = A/2B$, typically 2.8 m^{-2}.

For any particular flow less than the maximum, there are two possible states of the speed of pedestrians in the flow. (All pedestrians in the same area will move with the same speed.) The fast moving, low-density flow that we call the supercritical flow and the slow moving, high-density flow that we call the subcritical flow. (This notation is consistent with that used in mechanics but at variance with some works in traffic flow.) We refer to the different states as conjugate states. The behavior is identical to that of one-dimensional traffic flow as discussed for instance by Lighthill and Whitham (1955) and Whitham (1974).

It is useful to consider the application of (3.4) in a two-dimensional situation. Fig. 1 diagrammatically shows a curved passage of constant width in which we consider the pedestrian flow, possibly, as a model of a region of a far more complicated flow. If $(r_\mathrm{O} - r_\mathrm{I})/n \ll 1$, where r_O and r_I are the outer and inner radii of the passage, respectively, the situation reverts locally to a straight passage. Assuming the flow is a function of the radius, r, but not angular position around the passage, Eqs. (2.5a) and (2.5b) can be put into polar form to yield

$$\frac{\partial}{\partial\theta}\left(\rho f(\rho)^2 \frac{\partial\phi}{\partial\theta}\right) = 0 \tag{3.5a}$$

$$f(\rho) = \left|\frac{\partial\phi}{r\partial\theta}\right|^{-1}, \tag{3.5b}$$

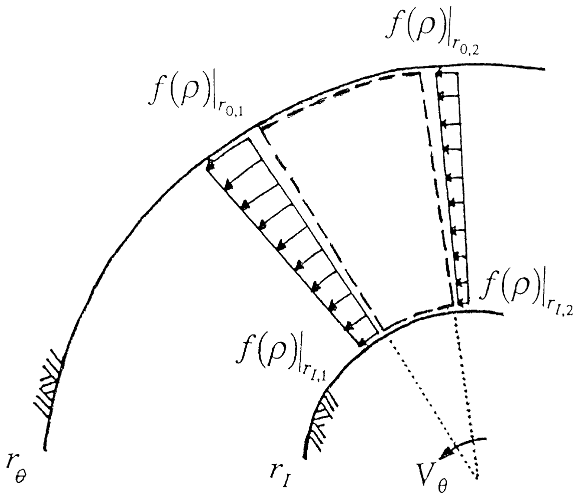

Fig. 1. Diagrammatic representation of flow in a curved passage showing the coordinate system used here. Also shown is a representation of a control volume moving along a curved passage with the speed (whatever that speed may be) of a disturbance to the pedestrian flow.

which integrates to give ρ as

$$\rho = \left(\frac{A - Cr}{B}\right), \tag{3.6}$$

where C is a positive constant of integration and A and B are as in (3.3). To determine C the boundary condition, $Q = \int_{r_I}^{r_O} f(\rho)\,\mathrm{d}r$ where Q is the total pedestrian flow, is applied. The result is the quadratic equation for C

$$|Q| = \frac{CA}{2B}\left[r_0^2\left(1 - \frac{2}{3}\frac{C}{A}r_0\right) - r_I^2\left(1 - \frac{2}{3}\frac{C}{A}r_I\right)\right]. \tag{3.7}$$

For large $|Q|$, as expected, there is no flow possible. For intermediate values of $|Q|$ there are two solutions for C, with the passage filled, corresponding to the subcritical and supercritical flows described earlier. For low $|Q|$, the supercritical flow is not bounded by a wall but by a curve of zero density. To show rigorously that the boundary condition at the free edge is one of zero density, suppose the density were non-zero at a free boundary. Then a pedestrian just within the flow would find finite advantage in leaving the flow and moving more quickly just outside the organized flow with infinitesimal extra distance of travel required. Thus for a free boundary

$$\rho = 0 \quad \text{at } Q = \int_{r_I}^{r_B} f(\rho)\,\mathrm{d}r, \tag{3.8}$$

where r_B is the value of the radius at the free edge of the flow.

To develop an appreciation of supercritical and subcritical regimes for a flow of pedestrians, it is useful to consider the form of Eqs. (2.5a) and (2.5b) at low flows. In the first case, that of a supercritical flow, spatial variations in ϕ are much less than in ρ and so the steady form of (2.5a) can be written as

$$\frac{\partial \rho}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \rho}{\partial y}\frac{\partial \phi}{\partial y} \approx 0, \tag{3.9}$$

which implies that lines of constant density are almost perpendicular to lines of constant potential. The density of the flow varies slowly in the direction of the flow. Away from boundaries, the edges of the flow are composed of straight-line segments with critical conditions, and hence a specified width, at those locations where the flow is forced to change direction. The flow predicted is very direct in its movement from the entrance to the exit.

In the second case, when the flow is subcritical, spatial variations in ρ are small compared to those in ϕ. In this case (2.5a) yields Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \approx 0, \tag{3.10}$$

with the normal derivative of ϕ equal to 0 at any barrier. In practical situations this equation can be conveniently solved numerically using relaxation (Southwell, 1946), say, or more quickly, considering the low level of accuracy likely to be required, by constructing the flow-net graphically (Taylor, 1948). The flow is less direct than in supercritical case and extends over the entire domain.

Interestingly, in Toshiyuki (1993) published account of a discrete model, where model pedestrians attempt to move along predefined globally determined paths, the flow pattern chosen by the modeller loosely lies between the above cases. Clearly, in this case the modeler has produced a compromise between the two flow regimes. However, as it is generally known whether the flow is supercritical or subcritical, a more accurate grid of the model directions can be easily obtained objectively. Depending on the situation, the modeler's subjectively obtained flow directions could lead to significant apparent errors maybe of the order of 25% in the time of travel. The simple procedure outlined here, which depends on the flow type, is of significant value and is fast to implement. It should be remembered that the path taken by a crowd is the net result of individuals in the crowd subjectively choosing a path. As such the situation has similarities to the discrete modeler choosing paths for individual pedestrians. However, the perspective of the modeler is different to that of the individuals in the crowd.

The names subcritical and supercritical have been given to the high- and low-density flows, respectively. These names are normally associated with the way in which disturbances propagate within the flow. Traditionally subcritical means that disturbances can propagate against the flow, while supercritical means that disturbances are swept along by the flow. It is necessary to establish that the names that have been chosen for the different regimes here are indeed consistent with the traditional nomenclature. To do this we consider a steady flow of pedestrians along a curved passage of locally constant width and curvature as considered earlier. A small amplitude, long disturbance is introduced to the flow by, say, slowly varying the number of pedestrians entering (or leaving depending on the control) the passage. Fig. 1 shows a control volume, bounded by the walls and two Sections 1 and 2 that move with the disturbance at angular speed V_θ. It follows from conservation of pedestrians in the control volume that the number of pedestrians entering is equal to the number leaving. Thus

$$\int \rho_1(f(\rho_1) - rV_\theta)\,\mathrm{d}r = \int \rho_2(f(\rho_2) - rV_\theta)\,\mathrm{d}r, \tag{3.11}$$

where ρ_1 and ρ_2 denote the densities at Sections 1 and 2, respectively, and the integrals are taken across the passage.

Solving for V_θ gives

$$V_\theta = \frac{\int \rho_2 f(\rho_2) - \rho_1 f(\rho_1)\,\mathrm{d}r}{\int r(\rho_2 - \rho_1)\,\mathrm{d}r} = \frac{\mathrm{d}\int \rho f(\rho)\,\mathrm{d}r}{\mathrm{d}\int r\rho\,\mathrm{d}r}, \tag{3.12}$$

in the limit of a small disturbance i.e. $\rho_1 \to \rho_2$.

For the low-density flow, disturbances propagate downstream in accordance with the usual understanding of a supercritical flow. Also, for the high-density flow, disturbances propagate upstream in accordance with the usual understanding of a subcritical flow. Control on the flow is achieved from wherever possible long waves originate. Thus, supercritical flow control is achieved from upstream, while subcritical flow control is achieved from downstream. This behavior is analogous to the subcritical conditions that extend upstream and supercritical conditions that extend downstream of traffic lights in vehicular flow. Thus (3.9) approximately describes the flow when upstream control exists and (3.10) describes it when the control is downstream.

It is interesting to note that Bradley (1993) saw waves, that by his account, moved in all directions. However, as noted earlier, he was considering a flow for which $f(\rho) \equiv 0$, that is, the flow was at rest except for wave motion. If we consider a flow of sufficiently high density that $f(\rho) = 0$ (3.12) yields, for a narrow passage in comparison to its radius which is undefined

$$V = \rho \frac{\mathrm{d}f(\rho)}{\mathrm{d}\rho}, \tag{3.13}$$

where V is the speed of the wave, rV_θ. As ρ and $\mathrm{d}f(\rho)/\mathrm{d}\rho$ are both finite (indeed numerically large) and nearly uniform, the speed V is finite and opposed to the direction of flow. As $f(\rho) = 0$ this can be any direction. However, the present model is only valid when the wave velocity amplitude is less than the velocity of the flow. So, despite the apparent descriptive agreement of the present model with the character of Bradley's (1993) waves, the model is invalid except for zero amplitude waves. The present study may be extended to very high-density flows in a manner consistent with Bradley (1993). Such an extension is outlined in Appendix A.

The stability of disturbances can also be inferred from a consideration of the relationship between q and ρ. As the addition of a disturbance changes the density, it also changes the speed of any additional disturbance to the flow. Furthermore, as

$$\frac{\mathrm{d}V_\theta}{\mathrm{d}\int r\rho\,\mathrm{d}r} = \frac{\mathrm{d}^2\int \rho f(\rho)\,\mathrm{d}r}{\mathrm{d}\int r\rho\,\mathrm{d}r^2} < 0, \tag{3.14}$$

using (3.3) and (3.6), a front associated with a decrease in density in the upstream direction will sharpen with time as additional disturbances upstream catch up to the initial disturbance. Such conditions occur for both subcritical and supercritical flow. A front with an increase in density upstream is always unstable. An unstable front will smooth with time to yield only gradual changes over large times.

4. Extension to pedestrians of multiple types

In Section 2 the partial differential equations governing the flow of pedestrians of a single type were deduced. Essential to that derivation were three hypotheses on pedestrian behavior. Often crowds consist of multiple pedestrian types where these types represent such diversity as differences in walking characteristics and destination. (Pedestrians of different origins may be of the same type.) Observations reported by Toshiyuki (1993) and reproduced here as Fig. 2 suggest a new first hypothesis.

Hypothesis 1A. The speed of pedestrians of a single type in multiple type flow is still determined by the function $f(\rho)$ but where ρ is now the total density rather than the density of a single pedestrian type.

There is a large body of literature supporting this somewhat surprising hypothesis. Its accuracy is apparent in some of the earliest quantitative observations on pedestrian flows such as those studied by Fruin (1971). However, it has been largely ignored in English language literature. A notable exception is the work of Al Gadhi and Mahmassani (1991) who thought that such a result was reasonable for very dense crowds and used it in a simulation after only casual observations. Ando et al. (1988) gave a simple explanation for this observed behavior, as represented in Fig. 3, for two pedestrian types with different destinations. They noted that pedestrian crowds form into bands as they pass through each other such that any pedestrian is walking amongst pedestrians of

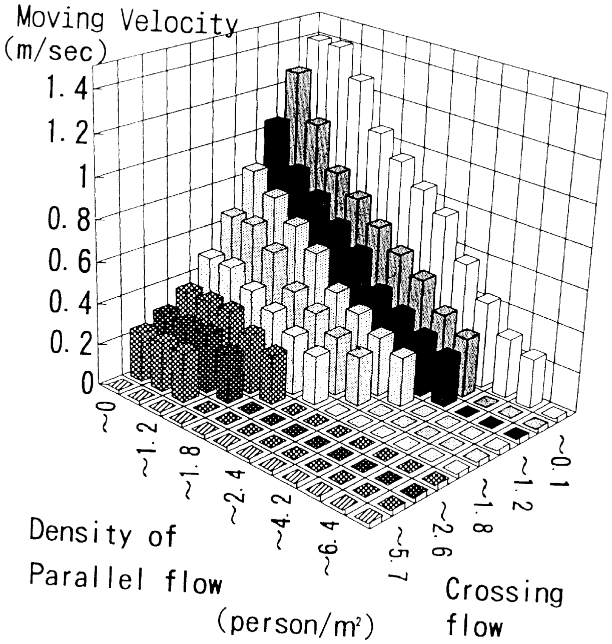

Fig. 2. Plot of pedestrian speed as a function of flow density and cross-flow density. Figure taken from Toshiyuki (1993).

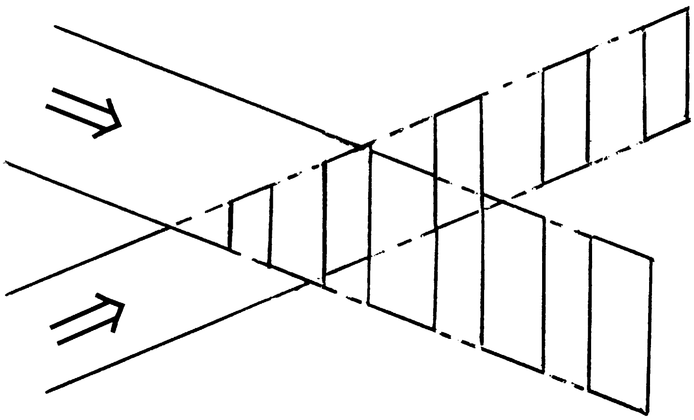

Fig. 3. Diagrammatic representation of how two flows pass through one another. Edited figure taken from Ando et al. (1988).

the same type but at a density equal to the combined density. There are delays at both entry and exit where the crowd must adjust. However, these delays are inconsequential if the delays are small in comparison with the crossing time. It is interesting to consider special situations where the destinations of the two pedestrian types are the same, but their walking characteristics are different. Observations by the author suggest that under such conditions the structure of the slower moving type pedestrian crowd adjusts to allow the faster type to pass freely provided the densities of the two crowds are similar. Thus, as might be expected, the hypothesis also holds in this case. If more than two pedestrian types are involved the argument given by Ando et al. (1988) still holds but the pattern of pedestrians is more complicated with bands within bands and therefore the pattern is presumably less stable.

The above hypothesis is supplemented by two further hypotheses analogous to the second and third hypotheses given earlier.

Hypothesis 2A. A potential field exists for each pedestrian type such that pedestrians move at right angles to lines of constant potential.

Hypothesis 3A. Pedestrians seek the path that minimizes their (estimated) travel time, but temper this behavior to avoid extremely high densities.

These latter two hypotheses follow directly from the discussion earlier and are not discussed further here.

Thus the flow of a particular type, type i, of pedestrians is given by $\rho_i f_i(\rho)$, where ρ_i is the density of pedestrians of a particular type and $f_i(\rho)$ is a specified function of total pedestrian density for pedestrians of type i. Note

$$\rho = \sum_{i=1}^{N} \rho_i, \tag{4.1}$$

where N is the number of pedestrian types. The governing equations, for each type i of pedestrian, are

$$-\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}\left(\rho_i g(\rho) f_i^2(\rho) \frac{\partial \phi_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_i g(\rho) f_i^2(\rho) \frac{\partial \phi_i}{\partial y}\right) = 0 \tag{4.2}$$

and

$$g(\rho) f_i(\rho) = \frac{1}{\sqrt{\left(\frac{\partial \phi_i}{\partial x}\right)^2 + \left(\frac{\partial \phi_i}{\partial y}\right)^2}}, \quad i = 1, \ldots, N, \tag{4.3}$$

where ϕ_i is the potential for type i of pedestrian moving over the (x, y) floor plan.

As noted earlier, a pedestrian type is determined by destination (not origin) as well as personal characteristics associated with walking speed and perception. Commonly, destinations are discrete. Pedestrians make their way to one of a finite number of objectives. However, personal characteristics are commonly a continuum. It is convenient to work with (4.2) and (4.3) by collecting pedestrians into groups (often called tribes) of approximate type rather than work with these equations in an integro-differential form.

In Section 3 it was shown that pedestrian flows of a single type can exist as either a subcritical or supercritical flow. The detailed form of $f(\rho)$ does not change the qualitative nature of the conjugate states. Here it is convenient, again, to consider a form of $f(\rho)$ that is linear in ρ. Thus, we write, for consistency with (3.3)

$$f_i(\rho) = \beta_i(A - B\rho), \quad i = 1, \ldots, N, \tag{4.4}$$

where β_i is of order unity but is different for each pedestrian type. Note that A and B are chosen as the same for all pedestrian types, so that all types pedlock at the same pedestrian density A/B. (It is conceivable that extremely aggressive pedestrians may be able to move at a slightly higher density but this effect is likely to be negligible in most situations and is ignored here.)

Consider a steady flow per unit width of each of N types of pedestrians, type $1, \ldots, N$, as given by

$$\begin{aligned} q_1 &= \rho_1 f_1(\rho) \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ q_N &= \rho_N f_N(\rho), \end{aligned} \tag{4.5}$$

respectively. Here $\rho_1, \ldots, \rho_N$ are the densities of the pedestrians in each of the N types such that

$$\rho = \sum_{i=1}^{N} \rho_i \tag{4.6}$$

and $f_1, \ldots, f_N$ are functions of form (4.4). Thus, dividing each q_i in (4.5) by β_i and adding yields

$$q_{\text{eff}} = \sum_{i=1}^{N} \frac{q_i}{\beta_i} = \rho(A - B\rho), \tag{4.7}$$

where the left-hand side is known as the effective flow. (Note that

$$q_{\text{eff}} \neq \sum_{i=1}^{N} q_i \tag{4.8}$$

however, all β_i are of order unity and q_{eff} is of the order of the total flow.) Hence from (4.7)

$$\rho = \frac{A}{2B} \pm \left(\left(\frac{A}{2B} \right)^2 - \frac{q_{\text{eff}}}{B} \right)^{1/2} \tag{4.9}$$

and if the effective flow is less than $A^2/4B$ (typically a modest 2/ms) two solutions exist (one subcritical and one supercritical). Critical conditions exist when the effective flow is $A^2/4B$ and no solution exists if the effective flow exceeds this critical value. Corresponding to any value of ρ, (4.5) implies unique values for $\rho_1, \ldots, \rho_N$ equal to $q_1/f_1(\rho), \ldots, q_N/f_N(\rho)$. Thus despite being N flows $q_1, \ldots, q_N$ there are only two possible states of the system not $2N$ as might be expected.

It is necessary to understand the motion of disturbances in a pedestrian flow as described here. Consider for simplicity two pedestrian types, moving at an angle, with uniform conditions both before and after a wave passes. Following Section 3, continuity of pedestrians yields the speed of the disturbance propagating in the direction of motion of the first pedestrian type as

$$V_1 = \frac{\mathrm{d}}{\mathrm{d}\rho_1}(\rho_1 f_1(\rho_1 + \rho_2)) \tag{4.10}$$

with ρ_2 constrained to vary with ρ_1 according to

$$\rho_2 f_2(\rho_1 + \rho_2) = \text{constant}. \tag{4.11}$$

We now wish to establish that if $V_1 = 0$ then $V_2 = 0$, that is, disturbances do not propagate in the direction of the motion of the second pedestrian type.

Setting $V_1 = 0$, (4.10) implies

$$\rho_1 f_1(\rho_1 + \rho_2) = \text{constant}. \tag{4.12}$$

Furthermore differentiating (4.11) with respect to ρ_2 subject to (4.12) implies

$$V_2 = \frac{\mathrm{d}}{\mathrm{d}\rho_2}(\rho_2 f_2(\rho_1 + \rho_2)) = 0. \tag{4.13}$$

Hence, if the flow is critical with respect to disturbances propagating in one direction it is critical with respect to disturbances propagating in the other direction.

It is clear that for high ρ_2 critical conditions, at which both V_1 and V_2 are zero, occur for a low value of ρ_1. Increasing the value of ρ_1 decreases the value of ρ_2 at which critical conditions occur. For lower values of ρ_1 than this critical value the flow is supercritical, while for higher values of ρ_1 the flow is subcritical. (Note that the terms subcritical and supercritical refer to the component of propagation in the direction of the flow.) The behavior is similar to that found in Section 3 when investigating the special case $\rho_2 \equiv 0$. A flow can be either subcritical or supercritical. It cannot be subcritical to one pedestrian type and supercritical to another.

5. Model of Jamarat bridge

The situation chosen for modeling here is that of the Jamarat Bridge in Mina, 5 km from Mecca in Saudi Arabia. In 2000, 2.1 million pilgrims crossed the bridge on the last day of the Hajj. It was the site of three major pedestrian disasters during the 1990s. In 1998, 118 pilgrims, in 1994, 250 pilgrims and in 1990, 1426 pilgrims died on the bridge from accidents. (To give perspective, a search of the news outlets shows that generally about a thousand people per year perish world-wide from excessive crowd density. About half of these fatalities are by crushing (asphyxiation) such as in the Danish disaster of June 2000 where 9 died. The other fatalities are by trampling (percussion) such as in the Zimbabwe disaster of July 2000 where 13 died.)

Pedestrians, that is pilgrims, flow over this bridge, on one of two levels, towards three approximately circular barriers of radius 8 m. Within each barrier stands one of three pillars, which the pedestrians stone, before moving on to the next barrier. The entire procedure is a symbolic re-enactment of Prophet Mohammed's stoning of the devil, who was, in turn, following the actions of Abraham. Selim and Al-Rabeh (1991) have modeled this pedestrian flow by adopting a systems approach, which represents the general flow from one pillar to another but not the detailed structure of that flow. In an excellent study, Al Gadhi and Mahmassani (1991) numerically simulated the crowd of pedestrians using quasi-axial symmetry about a single pillar. The results of aerial photographs, Anonymous (1984), described by Selim and Al-Rabeh (1991), show that pedestrians do spread around each of the pillars but concentrate on the upstream sides as illustrated in Fig. 4(a). The present study is directed at the distribution around a pillar and the analytical techniques and assumptions needed to study it.

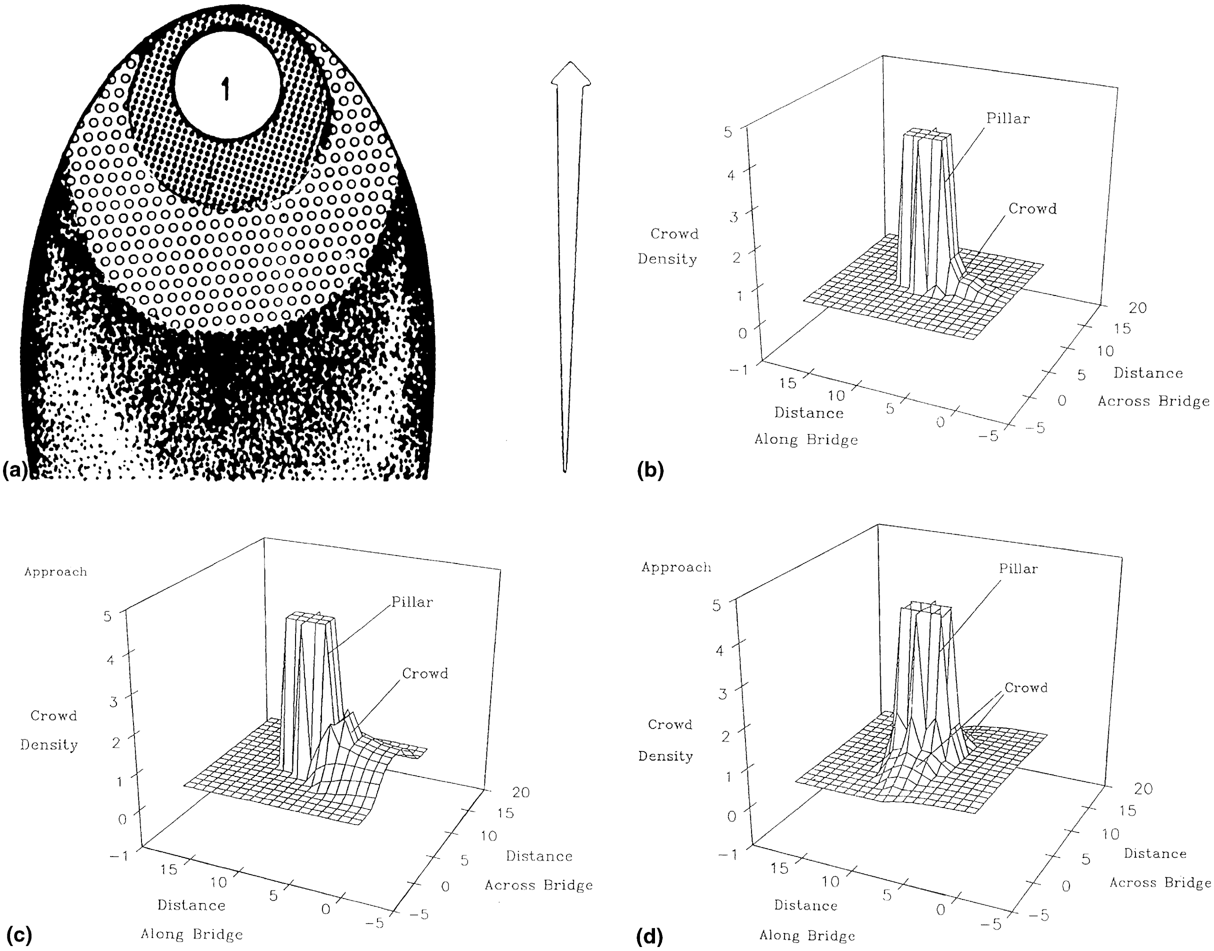

Fig. 4. The distribution of pedestrians around a pillar on the Jamarat Bridge. Shown are (a) the observed distribution based on aerial photographs from Selim and Al-Rabeh (1991), (b) the predicted distribution at low flow assuming pedestrians only approach the pillar, (c) the approaching distribution of pedestrians allowing for pedestrians leaving the pillar where the pillar is an equipotential and (d) the approaching distribution of pedestrians where the approach to the next pillar is the equipotential. Cases (c) and (d) have the same upstream potential as (b) which was chosen so that where the maximum density at the column is a modest 1 m^{-2}. The domain is square and extends across the bridge. The pillar diameter has been exaggerated in this study to be an exagerated 0.3 of the distance across the bridge. The values of A and B are 1.4 m/s and 0.25 m^3/s, respectively, throughout. Fig. 4(a) reprinted by permission, S.Z. Selim and A.H. Al-Rabeh, On the modeling of pedestrian flow on the Janarat bridge, Transp. Sci. 25, 257–262. © 1991, the Operations Research Society of America, currently the Institute for Operations Research and the Management Sciences, 901 Elkridge Landing Road, Suite 400, Linthicum, MD 21090–2909, USA.

As an approximation, let us assume that the pillars, which are approximately 150 and 190 m apart, can be studied in isolation. Here we model pedestrians approaching a single pillar, which is the objective to which they move, in such a way as to minimize their time of travel. By symmetry, it is only necessary to consider one side of the bridge in calculations using Eqs. (2.5a) and (2.5b). These calculations were carried out on a 21×11 rectangular grid that was used to construct a

21×21 grid across the bridge using a finite-difference method of integration. The lateral boundaries of the domain, corresponding to the side of the bridge and the center line are impenetrable and hence require $\partial\phi/\partial y = 0$. Fig. 4(b) shows the predicted steady pedestrian density, ρ, for a light pedestrian flow. The radius of the barrier around each pillar is $0.15\times$ the width of the bridge (i.e. 0.15×50 m ~ 8 m as described above). Here the retreat of pedestrians after stoning the pillar has been neglected. As such the flow is composed of pedestrians of a single type. Pedestrians spread around the pillar in the case of a heavy flow but for the light flow shown here pedestrians cluster on the approach side. It can be seen that the pattern resembles that displayed by Selim and Al-Rabeh (1991) from the aerial photographs. In the absence of detailed knowledge of the field situation it is difficult to be quantitative. The numbers given on the figure are those appropriate to the figure only and are only presumed to be of heuristic value in understanding the field situation. It should be noted that the density patterns observed do agree with those calculated here. For later work, in Sections 8–10, it should be noted that in the present section it is assumed that pedestrians must stand at the barrier to throw their stones. However, observations show that pedestrians often throw their stones from up to maybe ten people back from the barrier.

The previous calculation was not able to include accurately in the model the influence of pedestrians moving from a pillar on their way to the next. Such a flow is of two pedestrian types, those approaching and those leaving the pillar. Fig. 4(c) illustrates the density in the case of two pedestrian types. One type of pedestrian is that considered in Fig. 4(b), that is, pedestrians of uniform type approaching the pillar. The other type of pedestrians also of uniform type is that of pedestrians with the same β_i value but leaving the pillar. Here the barriers around the pillars are equipotentials. Of importance here is that the pedestrians leaving the pillar obstruct the pedestrians approaching the pillar and vice versa. This interaction of the two pedestrian types distorts the flow from that shown in Fig. 4(b). It is implicitly assumed that pedestrians are not driven by spontaneous alternation behavior, SAB, as described by Dember and Richman (1989). Such behavior might be important where pedestrians are fulfilling a lifetime dream of visiting Mecca. However, SAB is likely to be suppressed by the difficulty of movement in such a densely packed area.

In Fig. 4(c) it was assumed that the barrier around a pillar is an equi-potential. Pedestrian paths thus intersect the barrier at right angles. However, as pedestrians are intending to move to the next, for example second, pillar after stoning the first, the first barrier need not be an equipotential. It is only an equi-potential if it is the ultimate objective. If the first pillar is seen as part of an overall objective of stoning all three pillars then it is not an equi-potential. Fig. 4(d) illustrates the density if the first pillar is not seen in isolation. There is symmetry between the flows approaching and leaving the pillar. Stoning the forward side of the pillar requires a short walk to the first pillar followed by a long walk to the second. Stoning the aft side of the first pillar requires a long walk to the first pillar followed by a short walk to the second. The flow density given by Anonymous (1984) and reported by Selim and Al-Rabeh (1991) shows pedestrians see the pillars as separate objectives. Stoning the first pillar is of uppermost importance to them. They do not consider positioning themselves to stone the second pillar after the first. This behavior is maybe to be expected given the excitement and importance of the occasion. It may be possible to increase the flow of pedestrian flow over the bridge, or alternatively to increase pedestrian safety, by either directing or inducing pedestrians to the flanks and far side of any pillar. Such an inducement may be possible by making the barrier closer to the pillar there (or more distant elsewhere). Dr. Shokri Selim of King Fahd University of Petroleum and Minerals (per. com.) has stated that such distortions to the barrier shape have been tried already but there is still scope for further improvement of its shape. The 'optimal' barrier shape is discussed further in Section 9.

6. Alternative relationship for speed

Hypothesis 1 conveys the generally accepted idea that the speed of pedestrians is dependent in part on the density of pedestrians. Many such relationships have been proposed. Seven such relationships are compared by Virkler and Elayadath (1994). So far, we have taken the relationship for $f(\rho)$, the speed of pedestrians, as

$$f(\rho) = A(1 - \rho/\rho_{\max}), \tag{6.1}$$

using the analogy of pedestrian behavior with vehicular behavior, where B in (3.3) has been replaced by $A/\rho_{\max}$. This relationship is of limited accuracy. We consider an alternative form

$$\begin{aligned} f(\rho) &= A \quad \rho \leqslant \rho_{\text{trans}} \\ &= A(\rho_{\text{trans}}/\rho)^{1/2} \quad \rho_{\text{trans}} < \rho \leqslant \rho_{\text{crit}} \\ &= A\left(\frac{\rho_{\text{trans}}\rho_{\text{crit}}}{\rho_{\max} - \rho_{\text{crit}}}\right)^{1/2} \frac{(\rho_{\max} - \rho)^{1/2}}{\rho} \quad \rho_{\text{crit}} < \rho \leqslant \rho_{\max}, \end{aligned} \tag{6.2}$$

where A, ρ_{trans}, ρ_{crit} and $\rho_{\max}$ are the constants. The densities ρ_{trans}, ρ_{crit} and $\rho_{\max}$ have typical values of 0.8, 2.8 and 5.0 m^{-2}, respectively. The value of ρ_{crit} can be anywhere between ρ_{trans} and $\rho_{\max}$ but is generally closer to ρ_{trans}. Here it has been chosen for consistency with (3.3). As noted earlier, A is typically 1.4 m/s. Clearly, there are quantitative differences between (6.1) and (6.2), as seen in Fig. 5. However, the general character remains the same. This form was not considered by Virkler and Elayadath (1994), although, it has a shape similar to those curves considered superior to (6.1). In subsequent sections $f(\rho)$ is taken as given by (6.2). The form for the speed of pedestrians better captures the near constant speed at low densities and has the upward concave form commonly seen in observations. At ρ_{crit} there is an unrealistic discontinuity in the gradient of the speed.

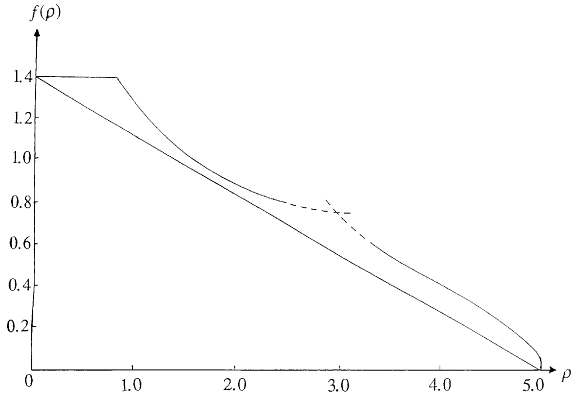

Fig. 5. Plot of pedestrian speed, $f(\rho)$, as a function of density, ρ, as given by (6.1) and (6.2) using values in the text.

However, with the extreme sensitivity of nearly critical flows it in unlikely that such a flow could ever be realized in practice and this pathological behavior is of no interest.

Pedestrians are often agitated in dense crowds where they fear for their own safety and consequently change their behavior as appears to occur dramatically in the observations of Hankin and Wright (1958). To model the desire to avoid such conditions the function $g(\rho)$ is taken as

$$\begin{aligned} g(\rho) &= 1 \quad \rho \leqslant \rho_{\text{crit}} \\ &= \frac{\rho(\rho_{\max} - \rho_{\text{crit}})}{\rho_{\text{crit}}(\rho_{\max} - \rho)} \quad \rho_{\text{crit}} < \rho < \rho_{\max}. \end{aligned} \tag{6.3}$$

For $\rho_{\text{crit}} < \rho < \rho_{\max}$, $g(\rho)$ is linear in ρ but as ρ tends to $\rho_{\max}$ (6.3) implies that $g(\rho)$ increases unboundedly. As will be shown in subsequent sections solving (4.2) and (4.3) with (6.2) and (6.3) it simple.

7. Conformal invariance of equations

Let us suppose that the region R with coordinates (x, y), over which pedestrians walk, is mathematically mapped into region W with coordinates (X, Y) by the conformal map

$$X + \mathrm{i}Y = H(x + \mathrm{i}Y), \tag{7.1}$$

where H is an analytic function at all but a finite number of points. Then transforming (4.2) and (4.3) with f given by (6.2), with $\rho_{\text{trans}} < \rho < \rho_{\text{crit}}$ from (x, y) coordinates to (X, Y) coordinates yields

$$-\frac{\partial R}{\partial t} + \frac{\partial}{\partial X}\left(RGF^2 \frac{\partial \phi}{\partial X}\right) + \frac{\partial}{\partial Y}\left(RGF^2 \frac{\partial \phi}{\partial Y}\right) = 0, \tag{7.2}$$

$$GF = \frac{1}{\sqrt{\left(\frac{\partial \phi}{\partial X}\right)^2 + \left(\frac{\partial \phi}{\partial Y}\right)^2}}, \tag{7.3}$$

where

$$R = |H'^2|^{-1}\rho, \tag{7.4}$$

$$F = |H'^2|^{1/2} f \tag{7.5}$$

and

$$G = g. \tag{7.6}$$

Clearly, (4.2) and (4.3) are the same as (7.2) and (7.3) with the following substitutions $x \to X, y \to Y$, $\rho \to R, f \to F, g \to G$. Thus, if a region is mapped from (x, y) space into (X, Y) space by a map of the form (7.1), a conformal map, the potential $\phi(x, y)$ remains unchanged. The variables $\phi(X, Y)$ and the density and speed similarly transform but with a scaling of $|H'^2|^{-1}$ and $|H'^2|^{1/2}$, respectively. Hence any steady or time-dependent solution in (x, y) space can be inferred from a solution in (X, Y) space.

Note that this conformal invariance of the equations is only valid if $f(\rho)$ is given by (6.2) with $\rho_{trans} < \rho < \rho_{crit}$. However, as shown in Section 9, this often presents no difficulty as pedestrians generally avoid, if possible, regions where $f(\rho)$ is constant, and in Section 10 a simple transformation is presented for $\rho_{crit} < \rho < \rho_{max}$ to make the equations conformally invariant.

8. Supercritical flow around a pillar without separation

Consider the flow, in (x, y) space, of pedestrians past the circular barrier of unit radius. Such a model might represent the globally optimum flow of pedestrians around a pillar on the Jamarat Bridge. This statement presupposes that pedestrians involved in the stoning could be induced to throw their stones whilst still walking and they walk long the optimal path for stoning all pillars. In this case, a unit of distance is 8 m. (The Mosque of Abraham imbedded in the flow around the Kaaba in Mecca where six lives were reportedly reported lost in the 2000 Hajj also provides a useful application. Pilgrims use this mosque to count their circuits of the Kaaba. The flow behavior can be seen in the time-lapse photograph of part of the flow around the Kaaba in Mecca as shown in Stewart, 1980). The geometry applicable to the flow passed the barrier around the pillar is shown in idealized form in Fig. 6(a), with U denoting the constant far field speed. Writing

$$X + \mathrm{i}Y = (x + \mathrm{i}y) + \frac{1}{(x + iy)}, \tag{8.1}$$

the region is transformed into the region in Fig. 6(b) in (X, Y) space. Any one of a number of dictionaries of conformal maps can be consulted to obtain simple conformal maps, for example Zill and Cullen (1992) or Kober (1957). The now trivial steady solution to (7.2) and (7.3) in this region is

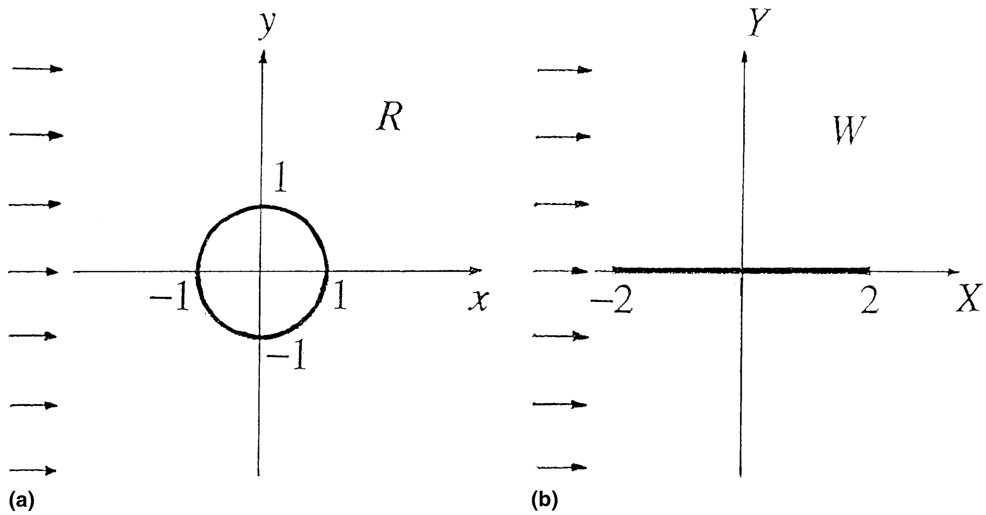

Fig. 6. Region in which the flow of pedestrians is to be solved in the study of flow around a column. Case (a) shows region in (x, y) space, and case (b) shows region in (X, Y) space.

$$\phi = -UX, \tag{8.2}$$

$$R = \left(\frac{C}{U}\right)^2, \tag{8.3}$$

$$F = U, \tag{8.4}$$

where the constant C is equal to $A\rho_{\text{trans}}^{1/2}$.

Thus transforming this solution from the (X, Y) space shown in Fig. 6(b) back to the (x, y) space shown in Fig. 6(a) using (8.1) yields

$$\phi = -Ux = \left(1 + \frac{1}{x^2 + y^2}\right), \tag{8.5}$$

$$\rho = \left(\frac{C}{U}\right)^2 \left|1 - \frac{1}{(x + \mathrm{i}y)^2}\right|^2, \tag{8.6}$$

$$f = U\left|1 - \frac{1}{(x + \mathrm{i}y)^2}\right|^{-1}. \tag{8.7}$$

Clearly on the circle $x^2 + y^2 = 1$, the normal derivative of ϕ (that is $x(\partial\phi/\partial x) + y(\partial\phi/\partial y)$) is zero, showing that as required there is no flow of pedestrians across the barrier. The solution for the flow density, ρ, is shown graphically in Fig. 7. (Note that the flow is around the barrier not towards the barrier (or pillar) as studied earlier.) The pedestrian density increases near the barrier.

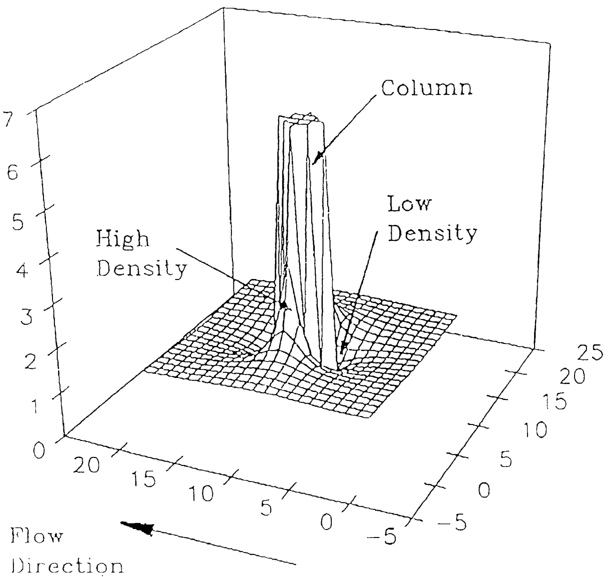

Fig. 7. Representation of the pedestrian density as the flow moves around the column of unit radius. The density has been scaled by $(C/U)^2$ as described in the text. The solution is evaluated as a 21×21 grid with a grid interval of 0.5 distance units. Note that although marked 'high' and 'low' density the entire flow has $\rho_{\text{trans}} < \rho \leqslant \rho_{\text{crit}}$ and is classified as a medium density flow.

Suppose now that the far field speed is not U, a constant, but $U(t)$ on the upstream boundary. A time varying $U(t)$ is consistent with the present management of the Hajj crowd. Presently, pedestrians are allowed onto the bridge in waves according to their nationality. Eq. (7.2) with (7.3) in (X, Y) space takes the form

$$-\frac{\partial R}{\partial t}+\frac{\partial}{\partial X}(RF)=0. \tag{8.8}$$

This first-order partial differential equation, see Sneddon (1957), has the solution

$$R=\left[\frac{C}{U\left(t-\frac{2R^{1/2}}{C}(X-X_i)\right)}\right]^2 \tag{8.9}$$

for $F = CR^{-1/2}$ as required by (7.2) and (7.3), and where X_i is the (assumed large negative) value of X at which $U(t)$ is specified.

Eq. (8.9) is an implicit expression for R. Associated with this expression for R,

$$F=U\left(t-\frac{2R^{1/2}}{C}(X-X_i)\right). \tag{8.10}$$

It follows from (7.4) and (7.5) that

$$\rho=R\left|1-\frac{1}{(x+\mathrm{i}y)^2}\right|^2, \tag{8.11}$$

$$f=F\left|1-\frac{1}{(x+\mathrm{i}y)^2}\right|^{-1} \tag{8.12}$$

with R and F given by (8.9) and (8.10) and where

$$X-X_i=x\left(1+\frac{1}{x^2+y^2}\right)-x_i, \tag{8.13}$$

with x_i, the value of x at which the boundary condition for the speed, known.

It can be seen that the pedestrian speed and density variations propagate into the region from upstream as a wave. If a sudden change in upstream conditions propagates into the region, the solutions downstream and upstream of this change correspond to the steady-state solutions with the initial and final states, respectively.

9. Supercritical flow around a pillar with separation

Immediately in front and behind the pillar studied in the previous section, Section 8, the pedestrian density was zero according to (8.6). In the adjacent regions it was finite but small and as such violated the use of (6.2) for which $\rho_{\text{trans}} < \rho \leqslant \rho_{\text{crit}}$. In cases where this occurs, pedestrians are predicted to travel at excessive speeds over large distances such that there is a finite value of ρ at every location. To accommodate this effect note that if the density ρ is predicted to be less than ρ_{trans} then it must be equal to zero because any pedestrian would not be able to move between equi-potentials quickly enough. (Equi-potentials are defined in terms of the time required to move from one equi-potential to another.)

The shape of the region devoid of pedestrians is unknown and to be determined here. This region is larger the less dense the pedestrian flow. Here the density is sufficiently low that pedestrians are only in contact with the barrier at the extreme sides of the barrier. Free streamline theory will be used as described by Batchelor (1970) for a fluid. Fig. 8 shows (a) physical (x, y) space containing a region of unknown shape both before and after (by symmetry) where no pedestrians walk, (b) shows (ϕ, ψ) space where ψ is defined by

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}, \tag{9.1}$$

as is possible for steady flow with $f(\rho)$ proportional to $\rho^{-1/2}$ as here, and (c) shows $(\ln(A/f(\rho)), \theta)$ space where θ is the angle of the pedestrian flow relative to its initial direction. Writing

$$z = x + \mathrm{i}y, \tag{9.2}$$

$$w = \theta + \mathrm{i}\psi, \tag{9.3}$$

and

$$\Omega = \ln(A/f(\rho)) + \mathrm{i}\theta, \tag{9.4}$$

it follows that:

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{1}{A}\mathrm{e}^{\Omega}. \tag{9.5}$$

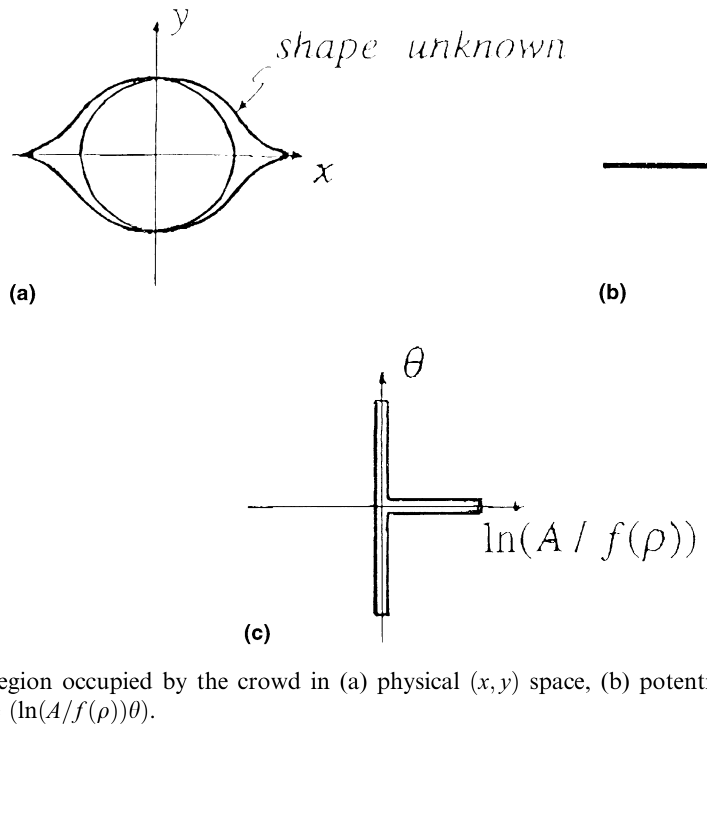

Fig. 8. The region occupied by the crowd in (a) physical (x, y) space, (b) potential (ϕ, ψ) space and (c) log inverse velocity space $(\ln(A/f(\rho))\theta)$.

See Batchelor (1970) for details.

The conformal map between Ω and w is given by inspection as

$$\Omega = C_1 \frac{w(w^2 - C_3^2)^{1/2}}{w^2 + C_2^2}, \tag{9.6}$$

where C_1 is a constant equal to $\ln(A/U)$ and C_2 and C_3 are constants seen later to be related to the diameter of the barrier. Substituting (9.6) into (9.5) and integrating with respect to w yields

$$z = \frac{1}{A} \int e^{C_1 \frac{w(w^2 - C_3^2)^{1/2}}{w^2 + C_2^2}} \, dw. \tag{9.7}$$

On the boundary between the separated region and the flow, where the speed of pedestrians is A, $|w| < C_3$. Hence, (9.7) can be written here in component form as

$$x = \frac{1}{A} \int_0^w \cos\left(C_1 \frac{w(C_3^2 - w^2)^{1/2}}{w^2 + C_2^2} \right) dw, \tag{9.8}$$

$$y = \frac{1}{A} \int_{-C_3}^w \sin\left(C_1 \frac{w(C_3^2 - w^2)^{1/2}}{w^2 + C_2^2} \right) dw, \tag{9.9}$$

where the terminals have been chosen on physical grounds.

If $(U - A)/U \ll 1$, as is the case for the separated region extending around all parts of the barrier but the extreme flanks, then $0 < \ln(A/U) \ll 1$ and so $C_1 \ll 1$. In this case, (9.8) and (9.9) yield

$$x \approx \frac{w}{A} \tag{9.10}$$

and

$$\begin{aligned} y &= \frac{1}{A} C_1 \int_{-C_3}^w \frac{w(C_3^2 - w^2)^{1/2}}{w^2 + C_2^2} \, dw, \\ &= \frac{1}{A} C_1 \left[(C_3^2 - w^2)^{1/2} - (C_2^2 + C_3^2)^{1/2} \operatorname{artanh}\left(\frac{C_3^2 - w^2}{C_3^2 + C_2^2} \right)^{1/2} \right]. \end{aligned} \tag{9.11}$$

It follows on eliminating w that the separated region is bounded by

$$y = \frac{\pm 1}{A} C_1 \left[(C_3^2 - A^2 x^2)^{1/2} - (C_2^2 + C_3^2)^{1/2} \operatorname{artanh}\left(\frac{C_3^2 - A^2 x^2}{C_3^2 + C_2^2} \right)^{1/2} \right]. \tag{9.12}$$

The ratio C_2/C_3 determine the maximum value of yA/C_1C_3 (which occur at $xA/C_3 = 0$). Thus changing C_3 changes the diameter of the barrier around which the pedestrians walk. We have C_3 is equal to A time the length of each separated region from the center of the column, $x_{\max}$. Thus it is convenient to write yA/C_1C_3 as $y/C_1 x_{\max}$, xA/C_3 as $x/x_{\max}$ and C_2/C_3 as $\hat{C}$ where $\hat{C} = C_2/Ax_{\max}$. The single non-dimensional parameter $\hat{C}$ is seen to require fitting according to the diameter of the column. Fig. 9 shows a plot of $y/C_1 x_{\max}$ as a function of $x/x_{\max}$ for two values of $\hat{C}$.

As $C = \ln(A/f) \ll 1$, y variations are small compared with x variations. Thus the separated region where no pedestrians walk is highly elongated along the x-axis. In the limit of $f(\rho)$ upstream being equal to A, that is light pedestrian flow, the region in front and behind the column, from the column to infinity, contains no pedestrians.

Clearly, if applied to the Jamarat Bridge, (9.12) provides the 'ideal' shape of the barrier upstream of a pillar. This shape provides the maximum deterrent to pedestrians taking the local view that each pillar is separate, without inhibiting pedestrians from following the path that clears pedestrians from the site most quickly. Such a shape ensures that the flow of pedestrians is discouraged from walking to the leading edge of the barrier and throwing their stones before moving to the next pillar. Instead, pedestrians can walk past the pillar, throwing their stones as they walk without any interference from the barrier. Note that there is practical engineering advantage in slightly rounding the leading edge of the calculated barrier shape. There is also advantage in not changing the barrier shape on the trailing edge. Leaving the barrier unchanged here assists those who have had trouble in throwing their stones while walking.

10. Subcritical flow around a pillar

In studies of high-density crowds, it is found that the flow of pedestrians decreases with increase in pedestrian density rather than increases as has been assumed in Sections 7–9. Such behavior is represented by (6.2) and (6.3) with the flow decreasing as $(\rho_{max} - \rho)^{1/2}$ for high-density flows. In this section, it will be shown that the equations of motion are also conformally invariant for such high-density flows. However, to achieve such invariance, the equations of motion must be written in terms of the density of unoccupied space (or holes) rather than the density of occupied space as considered earlier.

Hence, in studying high-density crowds, it is convenient to define the density of holes, σ, such that

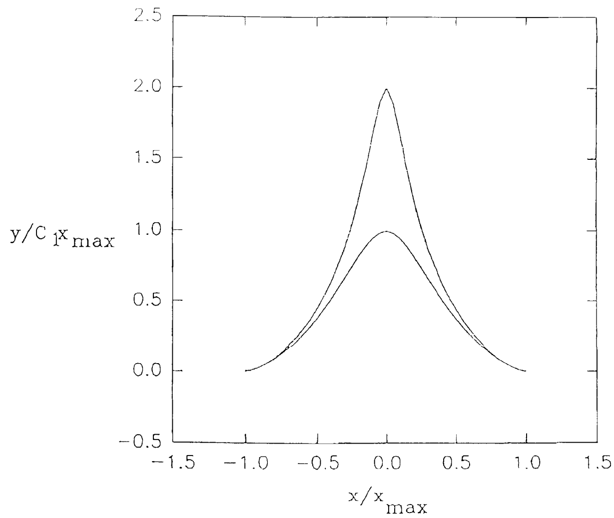

Fig. 9. The boundary of the flow. In non-dimensional form for $\hat{C}$ such that $y_{max}/C_1x_{max} = 1$ and 2 where $C_1 = \ln(A/f)$.

$$\sigma = \rho_{\max} - \rho \tag{10.1}$$

and the speed of holes, $h(\sigma)$, according to

$$\sigma h(\sigma) = \rho f(\rho). \tag{10.2}$$

Thus if $f(\rho)$ is given by (6.2) and (6.3) for $\rho_{\mathrm{crit}} < \rho \leqslant \rho_{\max}$,

$$h(\sigma) = A\left(\frac{\rho_{\mathrm{trans}} - \rho_{\mathrm{crit}}}{\rho_{\max} - \rho_{\mathrm{crit}}}\right)^{1/2}\sigma^{-1/2} \tag{10.3}$$

and (4.2) and (4.3) become

$$-\frac{\partial\sigma}{\partial t} + \frac{\partial}{\partial x}\left(\sigma h^2\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma h^2\frac{\partial\eta}{\partial y}\right) = 0 \tag{10.4}$$

and

$$h = \frac{1}{\sqrt{\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2}}, \tag{10.5}$$

provided $\eta = -\phi$. Eqs. (10.4) and (10.5) are the same equations as (4.2) and (4.3) with the following substitutions $\rho \to \sigma, f \to h, \phi \to \eta$. Furthermore with $h(\sigma)$ given by (10.3) the equations are again invariant under conformal mapping. Thus, the calculation of the path of a flow around a pillar in Section 8 remains valid without fear of flow separation. The path of such a flow is again given by (8.13).

In the case of the strategy developed earlier here for movement over the Jamarat Bridge, subcritical flows should be avoided. They lead to no additional flow and pose safety problems. However, if pedestrians stop to throw their stones, there is advantage in having a subcritical flow so that a large number of pedestrians are utilizing the throwing area simultaneously. The optimum flow regime in which a facility is operated can be influenced by barrier location within the facility.

11. Conclusions

Attention has been restricted to rational, goal-directed pedestrians. Any crowd may be divided into (approximate) pedestrian types where pedestrians in each type have the same walking habits. The entire study given here is based on the three hypotheses that imply that for each pedestrian type:

Hypothesis 1A. The speed of pedestrians of a single type in multiple type flow is determined by the function $f(\rho)$ but where ρ is the total density rather than the density of a single pedestrian type.

Hypothesis 2A. A potential field exists for each pedestrian type such that pedestrians move at right angles to lines of constant potential.

Hypothesis 3A. Pedestrians seek the path that minimizes their (estimated) travel time, but temper this behavior to avoid extremely high densities.

These hypotheses appear to be reasonable approximations to observed behavior. Hypothesis 1A is well established in the literature for a single pedestrian type. However, it appears to be valid also for multiple types. It appears to be only slightly restrictive although it is surprising. Hypothesis 2A assumes pedestrians are fully aware of their surroundings. It is conceivable that two tall, rational pedestrians may assess a situation totally differently from their different vantage points. Obstructions, for example, may be blocked from the view of one pedestrian but not from the view of the other. Under these conditions, the present theory fails. Hopefully however, the simplicity of the theory will ensure that it has value as an approximation even in cases where the second hypothesis is not strictly met. Hypothesis 3A is self-explanatory.

The present theory does not govern the behavior of any individual pedestrians. The theory is intended as a study of crowd behavior not individual behavior. Every individual has, of course, their own unique pedestrian characteristics with physical and mental attributes that distinguish them from all other pedestrians. They have been grouped into broader groups purely for analytical tractability.

The existence of multiple flow types, subcritical and supercritical flows with appropriate behavior of disturbances, is merely an extension of that encountered in modeling vehicular flows on roads (with a different notation). However, the two-dimensional nature of pedestrian flow contrasts to the one-dimensional nature of vehicular flow and does add some complications.

As illustrated with the Jamarat Bridge near Mecca, the behavior of pedestrians is not necessarily resolved by use of the model because the task to which the pedestrians respond is not always clearly defined. In the case of the Jamarat Bridge, the situation can be seen as either three separate tasks each involving the stoning of a single column or alternatively as a single task involving the stoning of the three columns. While these two views of the situation are the same in total, they have different expected pedestrian behavior. As shown here, pedestrians consider the situation as three separate tasks. The second alternative, of seeing the situation as a single task, gives hope for possibly increasing the capacity of the Jamarat Bridge. Such an increase may possibly be achieved by instruction of pedestrians before the site, guidance from ground marking or by a change in barrier shape. An 'optimal' barrier shape has been calculated here using simple conformal maps. Here 'optimal' is used to mean that the maximum possible discouragement has been given to pedestrians choosing a path that is optimized to the local goal rather than to the global goal. No direct influence is experienced by those choosing a globally determined path.

The solution of a pedestrian flow problem by the conformal mapping method described here is not limited to steady problems. No restriction exists on the form or amplitude of a time dependent disturbance in the flow. By the nature of the relationship between q and ρ used here, that is (3.1) and (3.3) or (6.2), all disturbances are either swept along by the pedestrian flow at medium densities or propagate against the flow at high densities.

The use of conformal mapping, as described in this study, provides a convenient method of analytically solving the equations for the pedestrian flow density and velocity field in any geometry. If an analytical solution is not required, as is common in practice, the direction of motion of the flow at all points in the domain can be determined rapidly. This may be done either by a graphical construction of the conformal flow net or, more commonly, numerically using relaxation. The present study provides a body of exact solutions, or the techniques to generate them, which can be used to validate a numerical model.

The method described here is a valuable tool for understanding, as opposed to merely simulating, pedestrian flows of a single type of pedestrian. It is appealing in its graphic nature. It is indeed remarkable that conformal mapping holds for the highly non-linear, time-dependent equations for pedestrian flow. It is even more remarkable that it can be used to study such flows at both medium and high densities, and to study the behavior of any discontinuity between them. It is hoped that the theory and its predictions will give a valuable means of interpreting numerical models and data.

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Appendix A. The equations of motion at very high density

The aim of this appendix is to bridge the gap between pedestrian models based on a fluid analogy and the present theory. The fluid analogy as explored experimentally by Bradley (1993) is only applicable at very high density.

It has been assumed in the Hypothesis 1 of the present study that

$$\varepsilon_x \equiv f(\rho)\hat{\phi}_x - u = 0 \text{ and } \varepsilon_y \equiv f(\rho)\hat{\phi}_y - v = 0, \tag{A.1}$$

where ε_x and ε_y are the errors involved in calculating u and v by using $f(\rho)$ for the speed. Thus, it was assumed that there is a strong force operating through the floor to the pedestrian flow, so that the left-hand side of (A.1) is kept zero. If pedestrians are moving too quickly for the density applicable to the area, the left-hand side of (A.1) is negative and a negatively directed force acts to slow the pedestrians down. Similarly, if the pedestrians are moving too slowly the left-hand side of (A.1) is positive and a positively directed force acts to increase their speed. For the right-hand side to be always zero, the force must be impulsive, that is unbounded in amplitude and infinitesimal in duration. This representation of the force may be unrealistic at the very high densities considered here.

At very high densities, this relationship needs modification because pedestrians may not be able to move at their desired speed and push on the floor in an effort to accelerate to their desired speed. Hence, we again modify Hypothesis 1. We concern ourselves with small values of ε, which we define as the maximum magnitude of the two quantities ε_x and ε_y defined in (A.1). Then using the first two terms of the Taylor expansion of (A.1), that is,

$$F_{fl} = F_{fl}\big|_{\varepsilon=0} + \frac{\partial F_{fl}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \varepsilon,$$

where F_{fl} and ε stands for F_{fl_x} or F_{fl_y} and ε_x or ε_y as is appropriate, leads to a new (double) Hypothesis 1, that is more complete than that given in the text.

Hypothesis 1B(i). The frictional force of the floor on pedestrians is proportional to the deficit in velocity (from that typical of pedestrians in a crowd of the same density). Thus we write this force, (F_{fl_x}, F_{fl_y}), as

$$F_{fl_x} = k(\rho)[f(\rho)\hat{\phi}_x - u] \quad \text{and} \quad F_{fl_y} = k(\rho)[f(\rho)\hat{\phi}_y - v], \tag{A.2}$$

where $k(\rho)$ is some function of density, and

Hypothesis 1B(ii). The force, (P_x, P_y), between pedestrians is normal and determined only by the gradient of pedestrian density. Thus, it can be written as

$$P_x = -\frac{\partial p(\rho)}{\partial x}, \quad P_y = -\frac{\partial p(\rho)}{\partial y}, \tag{A.3}$$

where $p(\rho)$ is some function relating the force between pedestrians (per unit horizontal distance) to the density.

Hypotheses 1B(i) and Hypotheses 1B(ii) replace Hypothesis 1 or Hypothesis 1A as appropriate. They are only applicable at very high-pedestrian densities and are justified here by nature that they lead to Bradley's (1993) formulation for very dense crowds. Thus for a dense crowd, in which the balance (A.1) has been disturbed, the force balance (per unit floor area) in the x and y directions, respectively, is

$$\begin{aligned} \rho\frac{\partial u}{\partial t} &= -\frac{\partial p(\rho)}{\partial x} + k(\rho)\Big[f(\rho)\hat{\phi}_x - u\Big], \\ \rho\frac{\partial v}{\partial t} &= -\frac{\partial p(\rho)}{\partial y} + k(\rho)\Big[f(\rho)\hat{\phi}_y - v\Big]. \end{aligned} \tag{A.4}$$

The term on the left-hand side represents the force needed to accelerate the crowd when the velocity is small (so that the advective acceleration is negligible). The first term on the right-hand side represents the net force between pedestrians. The second term represents the force of the floor on pedestrians. In the absence of better observations, the simplest suitable functions for $k(\rho)$ and $p(\rho)$ are

$$k(\rho) = D(\rho_c - \rho), \quad p(\rho) = \frac{E}{\rho_c - \rho}, \tag{A.5}$$

where D and E are constants. Note the form of (A.5) is slightly arbitrary just as the form of $f(\rho)$ and $g(\rho)$ in the text. The above equations must be combined with those originating from Hypotheses 2 and 3 to yield a complete description.

Bradley's (1993) equations are slightly more complex than (A.4) largely because he has included terms relating to a shear force between pedestrians. The influence of such terms depends strongly on their parameterisation. However, these shear force terms are likely to be small in practice, as they did not dampen the waves observed by Bradley (1993). As their importance is questionable, they are neglected here.

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